الحلول المتزامنة للمعادلات الجبرية الخطية

Solutions of Simultanenous Linear Algebraic Equations

3. SOLUTION OF A LINEAR SYSTEM BY JACOBI METHOD OF ITERATION OR GAUSS-JACOBI METHOD

All the previous methods of solving systems of simultaneous linear equations involve a certain amount of fixed computation. Now we shall describe the iterative or indirect methods of solving the systems. Here the amount of computation depends on the degree of accuracy required.

But the iterative method is not applicable to all systems of equations. Let us explain the Gauss-Jacobi method in the case of three equations in three unknowns.

Consider the system of equations,

$$a_1x + b_1y + c_1z = d_1 \qquad ...(1)$$

$$a_2x + b_2y + c_2z = d_2 \qquad ...(2)$$

$$a_3x + b_3y + c_3z = d_3 \qquad ...(3)$$

Suppose in the above equations

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|.$$

Then the Gauss-Jacobi method can be used for the given system. Solving equations (1), (2) and (3) for x, y and z respectively. That is,

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \qquad ...(4)$$

Suppose $x^{(0)}, y^{(0)}, z^{(0)}$ are the initial approximations of x, y, z respectively, then the first approximations given by

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)})$$

Again using values $x^{(1)}, y^{(1)}, z^{(1)}$ in (4), we have the second approximations given by

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1y^{(1)} - c_1z^{(1)})$$

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$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - c_3y^{(1)})$$

If we continue this process in the same way, the n^{th} approximations are $x^{(n)}, y^{(n)}, z^{(n)}$ and then

$$x^{(n+1)} = \frac{1}{a_1}(d_1 - b_1y^{(n)} - c_1z^{(n)})$$

$$y^{(n+1)} = \frac{1}{b_2}(d_2 - a_2x^{(n)} - c_2z^{(n)})$$

$$z^{(n+1)} = \frac{1}{c_3}(d_3 - a_3x^{(n)} - b_3y^{(n)})$$

This process is continued till the convergency is assured.

Note. In the absence of initial approximations $x^{(0)}, y^{(0)}, z^{(0)}$, they are taken as (0, 0, 0).

4. Gauss-Seidel Method of Iteration

This method is the most commonly used iterative method. This is a modification of Gauss-Jacobi method. As before, the equations can be solved for x, y, z respectively, we have

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

Now we can start the solution process with the initial values $x^{(0)}, y^{(0)}, z^{(0)}$ for x, y, z respectively. We can calculate, $x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)})$. Then we use this new value $x^{(1)}$ of x along with $z^{(0)}$ for z in the second equation, to compute the new value of $y^{(1)}$, we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(0)})$$

Then we use $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation to compute the new value $z^{(1)}$ of z, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$$

Thus as soon as a new value for a variable is found, it is used immediately in the following equations.

If $x^{(n)}, y^{(n)}, z^{(n)}$ are the n^{th} iterates, then the $(n+1)^{\text{th}}$ iteration will be

$$x^{(n+1)} = \frac{1}{a_1}(d_1 - b_1y^{(n)} - c_1z^{(n)})$$

$$y^{(n+1)} = \frac{1}{b_2}(d_2 - a_2x^{(n+1)} - c_2z^{(n)})$$

$$z^{(n+1)} = \frac{1}{c_3}(d_3 - a_3x^{(n+1)} - b_3y^{(n+1)})$$

This process is continued until the convergence is assured.

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Remarks. 1. The condition for convergence of above iterative methods is given by the following rule : The process of iteration will converge if in each equation of the system, the absolute value of the largest co-efficient is greater than the sum of the absolute values of all the remaining co-efficients.

2. The percent relative error is determined according to

$$\varepsilon_a = \left| \frac{\text{(Current approximation)} - \text{(Previous approximation)}}{\text{Current approximation}} \right| \times 100\%$$

WORKED EXAMPLES

Example 1. *Solve the given system of equations by using Gauss-Jacobi method.*

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110.$$

Solution. In each equation of the given system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining co-efficients. So Gauss-Jacobi method cab be applied solving each equation for x, y, z respectively,

$$x = \frac{1}{27}(85 - 6y + z) \quad \text{...(1)}$$

$$y = \frac{1}{15}(72 - 2z - 6x) \quad \text{...(2)}$$

$$z = \frac{1}{54}(110 - x - y) \quad \text{...(3)}$$

We start the iteration with the initial values $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$ for x, y, z respectively. Then we have,

$$x^{(1)} = \frac{1}{27}(85 - 6y^{(0)} + z^{(0)})$$

$$= \frac{85}{27} = 3.14815$$

$$y^{(1)} = \frac{1}{15}(72 - 2z^{(0)} - 6x^{(0)}) = \frac{72}{15} = 4.8$$

$$z^{(1)} = \frac{1}{54}(110 - x^{(0)} - y^{(0)}) = \frac{110}{54} = 2.03704$$

Second Iteration

$$x^{(2)} = \frac{1}{27}[85 - (6 \times 4.8) + 2.03704] = 2.15693$$

$$y^{(2)} = \frac{1}{15}[72 - (2 \times 2.03704) - (6 \times 3.14815)] = 3.26913$$

$$z^{(2)} = \frac{1}{54}[110 - 3.14815 - 4.8] = 1.88985$$

Third Iteration

$$x^{(3)} = \frac{1}{27}[85 - (6 \times 3.26913) + 1.88985] = 2.49167$$

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$$y^{(3)} = \frac{1}{15}[72 - (2 \times 1.88986) - (6 \times 2.15693)] = 3.68525$$

$$z^{(3)} = \frac{1}{54}[110 - 2.15693 - 3.26913] = 1.93655$$

Fourth Iteration

$$x^{(4)} = \frac{1}{27}[85 - (6 \times 3.68525) + 1.93655] = 2.40093$$

$$y^{(4)} = \frac{1}{15}[72 - (2 \times 1.93655) - (6 \times 2.49167)] = 3.54513$$

$$z^{(4)} = \frac{1}{54}[110 - 2.49167 - 3.68525] = 1.92265$$

Fifth Iteration

$$x^{(5)} = \frac{1}{27}[85 - (6 \times 3.54513) + (1.92265)] = 2.43155$$

$$y^{(5)} = \frac{1}{15}[72 - (2 \times 1.92265) - (6 \times 2.40093)] = 3.58327$$

$$z^{(5)} = \frac{1}{54}[110 - 2.40093 - 3.54513] = 1.92692$$

Sixth Iteration

$$x^{(6)} = \frac{1}{27}[85 - (6 \times 3.58327) + 1.92692] = 2.42323$$

$$y^{(6)} = \frac{1}{15}[72 - (2 \times 1.92692) - (6 \times 2.43155)] = 3.57046$$

$$z^{(6)} = \frac{1}{54}[110 - 2.43155 - 3.58327] = 1.92565$$

Seventh Iteration

$$x^{(7)} = \frac{1}{27}[85 - (6 \times 3.57046) + 1.92565] = 2.42603$$

$$y^{(7)} = \frac{1}{15}[72 - (2 \times 1.92565) - (6 \times 2.42323)] = 3.57395$$

$$z^{(7)} = \frac{1}{54}[110 - 2.42323 - 3.57046] = 1.92604$$

Eighth Iteration

$$x^{(8)} = \frac{1}{27}[85 - (6 \times 3.57395) + 1.92604] = 2.42527$$

$$y^{(8)} = \frac{1}{15}[72 - (2 \times 1.92604) - (6 \times 2.42603)] = 3.57278$$

$$z^{(8)} = \frac{1}{54}[110 - 2.42603 - 3.57395] = 1.92593$$

Ninth Iteration

$$x^{(9)} = \frac{1}{27}[85 - (6 \times 3.57278) + 1.92593] = 2.42552$$

$$y^{(9)} = \frac{1}{15}[72 - (2 \times 1.92593) - (6 \times 2.42527)] = 3.57310$$

$$z^{(9)} = \frac{1}{54}[110 - 2.42527 - 3.57278] = 1.92596$$

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Tenth Iteration

$$x^{(10)} = \frac{1}{27}[85 - (6 \times 3.57310) + 1.92596] = 2.42546$$

$$y^{(10)} = \frac{1}{15}[72 - (2 \times 1.92596) - (6 \times 2.42552)] = 3.57300$$

$$z^{(10)} = \frac{1}{54}[110 - 2.42552 - 3.57310] = 1.92595$$

In the 9th and 10th iterations we find that the values x, y, z are same correct to 3 decimal places. Here we stop the iteration process.

$\therefore$ The values are $x = 2.425$, $y = 3.573$, $z = 1.925$.

Example 2. *Solve the given system of equations by using Gauss-Seidel method :*

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25.$$

Solution. Here $|20| > |1| + |-2|$

$|20| > |3| + |-1|$

$|20| > |2| + |-3|$.

So Gauss-Seidel method can be used.

From the given equations we have

$$x = \frac{1}{20}(17 - y + 2z) \qquad ...(1)$$

$$y = \frac{1}{20}(-18 - 3x + z) \qquad ...(2)$$

$$z = \frac{1}{20}(25 - 2x + 3y) \qquad ...(3)$$

First Iteration. Let the initial values of y and z be $y^{(0)} = 0$, $z^{(0)} = 0$ and use in equation (1), we get, $x^{(1)} = \frac{1}{20}[17 - y^{(0)} + 2z^{(0)}] = \frac{17}{20} = 0.8500$.

Putting $z^{(0)} = 0$ and the current value $x^{(1)}$ of $x = 0.85$ in equation (2),

$$y^{(1)} = \frac{1}{20}[-18 - 3x^{(1)} + z^{(0)}]$$

$$= \frac{1}{20}[-18 - (3 \times 0.85) + 0] = -1.0275$$

Putting $x = x^{(1)} = 0.85, y = y^{(1)} = -1.0275$ in equation (3),

$$z^{(1)} = \frac{1}{20}[25 - 2x^{(1)} + 3y^{(1)}]$$

$$= \frac{1}{20}[25 - (2 \times 0.85) + (3 \times (-1.0275))] = 1.0109.$$

Second Iteration

$$x^{(2)} = \frac{1}{20}[17 - y^{(1)} + 2z^{(1)}]$$

$$= \frac{1}{20}[17 - (-1.0275) + 2 \times 1.0109]$$

$$= 1.0025$$

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$$y^{(2)} = \frac{1}{20}[-18 - 3x^{(2)} + z^{(1)}]$$
$$= \frac{1}{20}[-18 - 3 \times 1.0025 + 1.0109] = -0.9998$$
$$z^{(2)} = \frac{1}{20}[25 - 2x^{(2)} + 3y^{(2)}]$$
$$= \frac{1}{20}[25 - 2 \times 1.0025 + 3 \times (-0.9998)] = 0.9998$$

Third Iteration

$$x^{(3)} = \frac{1}{20}[17 - y^{(2)} + 2z^{(2)}]$$
$$= \frac{1}{20}[17 - (0.9998) + 2(0.9998)] = 1.0000$$
$$y^{(3)} = \frac{1}{20}[-18 - 3x^{(3)} + z^{(2)}]$$
$$= \frac{1}{20}[-18 - 3(1) + 0.9998] = -1.0000$$
$$z^{(3)} = \frac{1}{20}[25 - 2x^{(3)} + 3y^{(3)}]$$
$$= \frac{1}{20}[25 - 2(1) + 3(-1)] = 1.0000$$

Hence the values are $x = 1, \quad y = -1, \quad z = 1.$

Example 3. *Solve the following system by Gauss-Jacobi and Gauss-Seidel methods.*

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = -3.$$

Solution. Here in each equation, the absolute value of the largest co-efficient is greater than the sum of the remaining co-efficients. So, the iteration process can be applied.

Gauss-Jacobi method

Solving the equations for x, y, z, we have

$$x = \frac{1}{10}(3 + 5y + 2z) \quad ...(1)$$
$$y = \frac{1}{10}(3 + 4x + 3z) \quad ...(2)$$
$$z = \frac{1}{10}(-3 - x - 6y) \quad ...(3)$$

First Iteration

Let the initial values be $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$. Using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}[3 + 5y^{(0)} + 2z^{(0)}]$$
$$= \frac{1}{10}[3 + 5(0) + 2(0)] = 0.3$$

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$$y^{(1)} = \frac{1}{10}[3 + 4x^{(0)} + 3z^{(0)}]$$
$$= \frac{1}{10}[3 + 4(0) + 3(0)] = 0.3$$
$$z^{(1)} = \frac{1}{10}[-3 - x^{(0)} - 6y^{(0)}]$$
$$= \frac{1}{10}[-3 - 0 - 6(0)] = -0.3$$

Second Iteration

Using the values $x^{(1)}, y^{(1)}, z^{(1)}$ in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{10}[3 + 5(0.3) + 2(-0.3)] = 0.39$$
$$y^{(2)} = \frac{1}{10}[3 + 4(0.3) + 3(-0.3)] = 0.33$$
$$z^{(2)} = \frac{1}{10}[-3 - (0.3) - 6(0.3)] = -0.51$$

Third Iteration

$$x^{(3)} = \frac{1}{10}[3 + 5(0.33) + 2(-0.51)] = 0.363$$
$$y^{(3)} = \frac{1}{10}[3 + 4(0.39) + 3(-0.51)] = 0.303$$
$$z^{(3)} = \frac{1}{10}[-3 - (0.39) - 6(0.33)] = -0.537$$

Fourth Iteration

$$x^{(4)} = \frac{1}{10}[3 + 5(0.303) + 2(-0.537)] = 0.3441$$
$$y^{(4)} = \frac{1}{10}[3 + 4(0.363) + 3(-0.537)] = 0.2841$$
$$z^{(4)} = \frac{1}{10}[-3 - 0.363 - 6(0.303)] = -0.5181$$

Fifth Iteration

$$x^{(5)} = \frac{1}{10}[3 + 5(0.2841) + 2(-0.5181)] = 0.33843$$
$$y^{(5)} = \frac{1}{10}[3 + 4(0.3441) + 3(-0.5181)] = 0.2822$$
$$z^{(5)} = \frac{1}{10}[-3 - (0.3441) - 6(0.2841)] = -0.50487$$

Sixth Iteration

$$x^{(6)} = \frac{1}{10}[3 + 5(0.2822) + 2(-0.50487)] = 0.340126$$
$$y^{(6)} = \frac{1}{10}[3 + 4(0.33843) + 3(-0.50487)] = 0.283911$$
$$z^{(6)} = \frac{1}{10}[-3 - (0.33843) - 6(0.2822)] = -0.503163$$

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Seventh Iteration

$$x^{(7)} = \frac{1}{10}[3 + 5(0.283911) + 2(-0.503163)] = 0.3413229$$

$$y^{(7)} = \frac{1}{10}[3 + 4(0.340126) + 3(-0.503163)] = 0.2851015$$

$$z^{(7)} = \frac{1}{10}[-3 - (0.340126) - 6(0.283911)] = -0.5043592$$

Eighth Iteration

$$x^{(8)} = \frac{1}{10}[3 + 5(0.2851015) + 2(-0.5043592)] = 0.34167891$$

$$y^{(8)} = \frac{1}{10}[3 + 4(0.3413229) + 3(-0.5043592)] = 0.2852214$$

$$z^{(8)} = \frac{1}{10}[-3 - (0.3413229) - 6(0.2851015)] = -0.50519319$$

Ninth Iteration

$$x^{(9)} = \frac{1}{10}[3 + 5(0.2852214) + 2(-0.50519319)] = 0.341572062$$

$$y^{(9)} = \frac{1}{10}[3 + 4(0.34167891) + 3(-0.50519319)] = 0.285113607$$

$$z^{(9)} = \frac{1}{10}[-3 - (0.34167891) - 6(0.2852214)] = -0.505300731.$$

The values correct to 3 decimal places are

$$x = 0.341, \quad y = 0.285, \quad z = -0.505.$$

Gauss-Seidel method. Taking the initial values $y^{(0)} = 0$, $z^{(0)} = 0$

First Iteration

$$x^{(1)} = \frac{1}{10}[3 + 5y^{(0)} + 2z^{(0)}] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3 + 4x^{(1)} + 3z^{(0)}] = 0.42$$

$$z^{(1)} = \frac{1}{10}[-3 - x^{(1)} - 6y^{(1)}] = -0.582$$

Second Iteration

$$x^{(2)} = \frac{1}{10}[3 + 5y^{(1)} + 2z^{(1)}] = 0.3936$$

$$y^{(2)} = \frac{1}{10}[3 + 4x^{(2)} + 3z^{(1)}] = 0.28284$$

$$z^{(2)} = \frac{1}{10}[-3 - x^{(2)} - 6y^{(2)}] = -0.509064$$

Third Iteration

$$x^{(3)} = \frac{1}{10}[3 + 5y^{(2)} + 2z^{(2)}] = 0.3396072$$

$$y^{(3)} = \frac{1}{10}[3 + 4x^{(3)} + 3z^{(2)}] = 0.28312368$$

$$z^{(3)} = \frac{1}{10}[-3 - x^{(3)} - 6y^{(3)}] = -0.503834928$$

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Fourth Iteration

$$x^{(4)} = \frac{1}{10}[3 + 5y^{(3)} + 2z^{(3)}] = 0.34079485$$

$$y^{(4)} = \frac{1}{10}[3 + 4x^{(4)} + 3z^{(3)}] = 0.285167464$$

$$z^{(4)} = \frac{1}{10}[-3 - x^{(4)} - 6y^{(4)}] = -0.50517996$$

Fifth Iteration

$$x^{(5)} = \frac{1}{10}[3 + 5y^{(4)} + 2z^{(4)}] = 0.34155477$$

$$y^{(5)} = \frac{1}{10}[3 + 4x^{(5)} + 3z^{(4)}] = 0.28506792$$

$$z^{(5)} = \frac{1}{10}[-3 - x^{(5)} - 6y^{(5)}] = -0.505196229$$

Sixth Iteration

$x^{(6)} = 0.341494714$

$y^{(6)} = 0.285039017$

$z^{(6)} = -0.5051728$

Seventh Iteration

$x^{(7)} = 0.3414849$

$y^{(7)} = 0.28504212$

$z^{(7)} = -0.5051737$

The values correct to 3 decimal places are

$$x = 0.342, \quad y = 0.285, \quad z = -0.505.$$

Example 4. *Solve the given system of equations by using Gauss-Jacobi method and Gauss-Seidel method*

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35.$$

Solution. Here $|8| > |-3| + |2|$

$$|11| > |4| + |-1|$$
$$|12| > |6| + |3|$$

So the iteration process can be used.

Solving for x, y, z, we get

$$x = \frac{1}{8}[20 + 3y - 2z] \quad ...(1)$$

$$y = \frac{1}{11}[33 - 4x + z] \quad ...(2)$$

$$z = \frac{1}{12}[35 - 6x - 3y] \quad ...(3)$$

Starting with the initial values $x = 0, y = 0, z = 0$ and using (1), (2), (3) and repeating the process we get the values of x, y, z as the tabulated by both Gauss-Jacobi and Gauss-Seidel methods.

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Iteration	Gauss-Jacobi Method			Gauss-Seidel Method		
	x	y	z	x	y	z
1	2.5	3.0	2.916666	2.5	2.090909	1.143939
2	2.895833	2.356060	0.916666	2.998106	2.013774	0.914170
3	3.154356	2.030303	0.879735	3.026623	1.982516	0.907726
4	3.041430	1.932937	0.831913	3.016512	1.985607	0.912009
5	3.016873	1.969654	0.912717	3.016600	1.985964	0.911876
6	3.010441	1.985930	0.915817	3.016767	1.985892	0.911810
7	3.015770	1.988550	0.914964	3.016757	1.985889	0.911816
8	3.016946	1.986535	0.911644			
9	3.017039	1.985805	0.911560			
10	3.016786	1.985764	0.911696			

Hence $x = 3.016,\quad y = 1.985,\quad z = 0.911.$

EXERCISES

1. Using Gauss-Jacobi method, solve the following systems of equations :

(i) $x + 17y - 2z = 48$
$30x - 2y + 3z = 75$
$2x + 2y + 18z = 30$ [Ans. 2.58, 2.798, 1.069]

(ii) $5x - 2y + z = -4$
$x + 6y - 2z = -1$
$3x + y + 5z = 13$ [Ans. – 1, 1, 3]

(iii) $8x - 6y + z = 13.67$
$3x + y - 2z = 17.59$
$2x - 6y + 9z = 29.29$ [Ans. 2.45, 1.62, 3.79]

(iv) $13a + 5b - 3c + d = 18$
$2a + 12b + c - 4d = 13$
$3a - 4b + 10c + d = 29$
$2a + b - 3c + 9d = 31$ [Ans. 2, 3, 4]

(v) $10x + y - z = 11.19$
$x + 10y + z = 20.08$
$-x + y + 10z = 35.61.$ [Ans. 1.321, 1.522, 3.541]

2. Using Gauss-Seidel method, solve the following systems of equations :

(i) $28x + 4y - z = 32$
$x + 3y + 10z = 24$
$2x + 17y + 4z = 35$ [Ans. 0.9936, 1.5069, 1.8486]

(ii) $1.2x + 2.1y + 4.2z = 9.9$
$5.3x + 6.1y + 4.7z = 21.6$
$9.2x + 8.3y + z = 15.2$ [Ans. – 13.223, 16.766, – 2.306]

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Interpolation with Unequal Intervals

1. INTRODUCTION

In the last chapter, we have discussed Newton's forward and backward interpolation formulae and Gauss's central difference formulae which are applicable only when the values of argument x are given at equal intervals.

If the values of x are given at unequal intervals, the above formulae can not be applicable. So it is convenient to introduce the idea of divided differences. The differences defined taking into consideration the changes in the values of the argument are called divided differences.

1.1 LAGRANGE'S INTERPOLATION FORMULA

The Newton's forward and backward interpolation formulae can be used only when the values of independent variable x are equally spaced. And the differences of y must become ultimately small. But in cases, where the values of independent variable x are not equally spaced and in cases when the differences of dependent variable y are not small, ultimately, we can not use Newton's formula.In such case we use Lagrange's interpolation formula.

Let the function $y = f(x)$ take the values $f(x_0), f(x_1), \ldots f(x_n)$ corresponding to the values x_0, $x_1, x_2, \ldots x_n$ of the argument x, where the x values are not equally spaced. Since there are $(n + 1)$ values of y corresponding to $(n + 1)$ values of x, we can represent the function $f(x)$ by a polynomial in x of degree n.

Now we select the polynomial $f(x)$ as follows:

$$\begin{aligned} y = f(x) &= a_0(x - x_1)(x - x_2)(x - x_3) \ldots (x - x_n) \\ &+ a_1(x - x_0)(x - x_2)(x - x_3) \ldots (x - x_n) \\ &+ a_2(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ a_{n-1}(x - x_0)(x - x_1)(x - x_2) \ldots (x - x_n) \\ &+ a_n(x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1}) \end{aligned} \quad \ldots(1)$$

In the R.H.S. of eqn. (1), the term in which a_i occurs has the factor $(x - x_i)$ lacking. Equation (1) is true for all values of x. Substituting $x = x_0, y = y_0$ in (1), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$\Rightarrow \quad a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)}$$

Similarly putting $x = x_1, y = y_1$ we have

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)}$$

In the same way we get

$$a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\dots(x_2 - x_n)}$$

....

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, \dots a_n$ in (1), we get

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)\dots(x_0 - x_n)} y_0$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\dots(x_1 - x_n)} y_1$$

$$+ \dots\dots\dots\dots$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)\dots(x - x_n)}{(x_{n-1} - x_0)(x_{n-1} - x_1)(x_{n-1} - x_2)\dots(x_{n-1} - x_n)} y_{n-1}$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)\dots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\dots(x_n - x_{n-1})} y_n \qquad \dots(2)$$

Equation (2) is called Lagrange's interpolation formula for unequal intervals.

Note 1. The linear version ($n = 1$)

$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$ is called first order Lagrange interpolation polynomial.

Note 2. The second order version ($n = 2$)

$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$ is called second order Lagrange interpolation polynon

WORKED EXAMPLES

Example 1. *Use Lagrange's interpolation formula to fit a polynomial to the data:*

x:	*– 1*	*0*	*2*	*3*
y:	*– 8*	*3*	*1*	*12*

and hence find y at x = 1.

Solution. Here the given data contains four pairs of values.

By Lagrange's formula,

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Here $x_0 = -1, x_1 = 0, x_2 = 2, x_3 = 3$

$y_0 = -8, y_1 = 3, y_2 = 1, y_3 = 12.$

$$y(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3) + \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)} 12$$

$$= \frac{8}{12}(x)(x-2)(x-3) + \frac{1}{2}(x+1)(x-2)(x-3) - \frac{1}{6}(x+1)x(x-3) + (x+1)x(x-2)$$

$$= 2x^3 - 6x^2 + 3x + 3 \qquad \text{(on simplification)}$$

$$y(x=1) = \frac{8}{12}(1)(1-2)(1-3) + \frac{1}{2}(1+1)(1-2)(1-3) - \frac{1}{6}(1+1)(1)(1-3) + (1+1)(1)(1-2)$$

$$= \frac{16}{12} + \frac{4}{2} + \frac{4}{6} - 2 = \frac{4}{3} + 2 + \frac{2}{3} - 2 = 2.$$

Example 2. *Use Lagrange's interpolation formula to find the value of f(x) corresponding to x = 27 from the following data:*

x :	*14*	*17*	*31*	*35*
f(x):	*68.7*	*64.0*	*44.0*	*39.1*

Solution. By Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Here $x_0 = 14, x_1 = 17, x_2 = 31, x_3 = 35$

$y_0 = 68.7, y_1 = 64.0, y_2 = 44.0$ and $y_3 = 39.1$

$$\therefore \quad y = f(x) = \frac{(x-17)(x-31)(x-35)}{(14-17)(14-31)(14-35)}(68.7) + \frac{(x-14)(x-31)(x-35)}{(17-14)(17-31)(17-35)}(64.0) + \frac{(x-14)(x-17)(x-35)}{(31-14)(31-17)(31-35)}(44.0) + \frac{(x-14)(x-17)(x-31)}{(35-14)(35-17)(35-31)}(39.1)$$

i.e.,

$$y = -\frac{(68.7)}{1071}(x-17)(x-31)(x-35) + \frac{64}{756}(x-14)(x-31)(x-35)$$
$$-\frac{44}{952}(x-14)(x-17)(x-35) + \frac{39.1}{1512}(x-14)(x-17)(x-31)$$

$$y(x=27) = -\frac{(68.7)}{1071}(27-17)(27-31)(27-35)$$
$$+\frac{64}{756}(27-14)(27-31)(27-35)$$
$$-\frac{44}{952}(27-14)(27-17)(27-35)$$
$$+\frac{39.1}{1512}(27-14)(27-17)(27-31)$$
$$= -20.52 + 35.22 + 48.07 - 13.45 = 49.3.$$

==

Example3: Use Lagrange's method to find log_{10} 656 give that log_{10} 656=2.8156, log_{10} 659=2.8189 and log_{10} 661=2.8202.

Solution: Putting x_0=654 ,x_1=658 ,x_2=659 ,x_3=661 and y_0=2.8156 y_1=208182

y_2=2.8189 ,y_3=2.8202 in the Lagrange's interpolation formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_4 \quad \ldots(1)$$

We have

$$\log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times (2.8156) + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-654)(658-661)} \times 2.8182 + \frac{(656-654)(656-658)(656-661)}{(658-654)(658-654)(658-661)} \times 2.8189 + \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$= \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} \times (2.8156) + \frac{(2)(-3)(-5)}{(4)(-1)(-3)} \times (2.8182) + \frac{(2)(-2)(-5)}{(5)(1)(-2)} \times 2.8189 + \frac{(2)(-2)(-3)}{(7)(3)(2)} \times (2.8202)$$

$$= 0.6033 + 7.0455 - 5.6378 + 0.8058 = 2.8168.$$

Example 4. *Find the equation of the cubic curve that passes through the points (– 1, – 8), (0, 3), (2, 1) and (3, 2) using Lagrange's interpolation formula.*

Solution. Putting $x_0 = -1, x_1 = 0, x_2 = 2, x_3 = 3$ and $y_0 = -8, y_1 = 3, y_2 = 1, y_3 = 2$ in the Lagrange's interpolation formula (1) given in example (4), we have

$$y = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3) + \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(2)$$

$$= \frac{2}{3}(x^3 - 5x^2 + 6x) + \frac{1}{2}(x^3 - 4x^2 + x + 6) - \frac{1}{6}(x^3 - 2x^2 - 3x) + \frac{1}{6}(x^3 - x^2 - 2x)$$

i.e.,

$$y = \frac{7}{6}x^3 - \frac{31}{6}x^2 + \frac{14}{3}x + 3$$

∴ The equation of the required cubic curve is $6y = 7x^3 - 31x^2 + 28x + 18$.

Example 5. *Use Lagrange's interpolating polynomial of the first and second order to the given data:*

x :	1	4	6
f(x):	0	1.3863	1.7918.

And find f(x) at x = 2.

Solution. Here $x_0 = 1, x_1 = 4, x_2 = 6$ and $y_0 = 0, y_1 = 1.3863, y_2 = 1.7918$

The first order Lagrange polynomial is

$$f_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

$$\Rightarrow \quad f_1(x) = \frac{(x-4)}{(1-4)}(0) + \frac{x-1}{(4-1)}(1.3863)$$

i.e., $f_1(x) = (0.4621)\,[x - 1]$

And $f_1(2) = (0.4621)\,(2 - 1) = 0.4621$

The second order Lagrange polynomial is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(\dots$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\dots$$

$$\Rightarrow \quad f_2(x) = \frac{(x-4)(x-6)}{(1-4)(1-6)}(0) + \frac{(x-1)(x-6)}{(4-1)(4-6)}(1.3863)$$

$$+ \dots\dots$$

$$= 0 - (0.2311)(x^2 - 7x + 6) + (0.1792)(x^2 \quad x + 4)$$

i.e., $f_2(x) = -0.0519x^2 + 0.7217x - 0.6698$

And $f_2(2) = (-0.0519 \times 4) + (0.7217 \times 2) - \;\;6698 = \;\;\;\;60.$

EXERCISES

1. Using Lagrange's formula find $f(6)$ given:

x :	2	5	7	10	12
$f(x)$:	18	180	448	1210	2028

[Ans. 294]

2. Using Lagrange's formula find $f(10)$ from the following table:

x :	5	6	9	11
$f(x)$:	12	13	14	16

[Ans. 14.66]

3. Using Lagrange's formula find y at $x = 6$ from the following data:

x:	3	7	9	10
y:	168	120	72	63

[Ans. 147]

4. Using Lagrange's formula find y at $x = 9.5$ given:

x:	7	8	9	10
y:	3	1	1	9

[Ans. 3.625]

5. The following data give the percentage of Criminals for the different age groups:

Age (less than x):	25	30	40	50
% of Criminals :	52.0	67.3	84.1	94.4

Using Lagrange's formula, estimate the percentage of Criminals under the age 35. [Ans. 77.4]

6. Given $y_1 = 22, y_2 = 30, y_4 = 82, y_7 = 106, y_8 = 206$ find y_6. [Ans. 83.515]

7. If $y_1 = 4, y_3 = 120, y_4 = 340, y_6 = 2544$ find y_5. [Ans. 1052]

Interpolation with Equal Intervals

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2. GREGORY–NEWTON FORWARD INTERPOLATION FORMULA OR NEWTON'S FORWARD INTERPOLATION FORMULA FOR EQUAL INTERVALS

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \dots y_n$ corresponding to the values $x_0, x_1, x_2, \dots x_n$ of the independent variable x.

Let the values of x be at equidistant intervals.

i.e., $$x_i - x_{i-1} = h,\ i = 1, 2, \dots n.$$

Then
$$x_1 = x_0 + h$$
$$x_2 = x_1 + h = x_0 + 2h$$

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$$x_3 = x_2 + h = x_0 + 3h \text{ etc.}$$
$$\therefore \quad x_i = x_0 + ih,\ i = 1, 2, \dots n.$$

Let $y(x)$ be a polynomial of the nth degree in x taking the same values as y_x corresponding to the x values $x_0, x_1, x_2, \dots x_n$.

i.e., $$y(x_i) = y_i \text{ for } i = 1, 2, \dots n$$

Now we want to find a collocation polynomial $P_n(x)$ of degree n in x such that
$$y(x_i) = P_n(x_i),\ i = 0, 1, 2, \dots n.$$

Let
$$y(x) = P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) \quad \dots(1)$$

This polynomial contains the $(n + 1)$ constants $a_0, a_1, \dots a_n$ which can be determined as follows:

Putting $x = x_0, x_1, x_2, \dots, x_n$ successively in (1), we get

$$y(x_0) = a_0 \ \Rightarrow\ y_0 = a_0 \quad \dots(2)$$
$$y(x_1) = a_0 + a_1(x_1 - x_0)$$
$$\Rightarrow \quad y_1 = a_0 + a_1 h \quad \dots(3)$$
$$y(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$\Rightarrow \quad y_2 = a_0 + a_1(x_2 - x_1 + x_1 - x_0) + a_2(x_2 - x_1 + x_1 - x_0)(x_2 - x_1)$$
$$\Rightarrow \quad y_2 = a_0 + 2ha_1 + 2h^2 a_2 \quad \dots(4)$$

and so on.

From these, the values of $a_0, a_1, a_2, \dots$ can be found in terms of y_0 and its forward differences.

Using (2) in (3), we have
$$y_1 = y_0 + a_1 h$$
$$\Rightarrow \quad a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

Equation (4) gives $$y_2 = y_0 + 2h \times \frac{\Delta y_0}{h} + 2h^2 a_2$$
$$\Rightarrow \quad y_2 - y_0 - 2\Delta y_0 = 2h^2 a_2$$
$$\Rightarrow \quad a_2 = \frac{y_2 - y_1 + y_1 - y_0 - 2\Delta y_0}{2h^2} = \frac{\Delta y_1 - \Delta y_0}{2h^2} = \frac{\Delta^2 y_0}{2!\,h^2}$$

Similarly $$a_3 = \frac{\Delta^3 y_0}{3!\,h^3} \text{ and so on.}$$

Putting these values in (1), we can have
$$y(x) = P_n(x) = y_0 + \frac{\Delta y_0}{h}.(x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\,h^3}(x - x_0)(x - x_1)(x - x_2) + \dots + \frac{\Delta^n y_0}{n!\,h^n}(x - x_0)(x - x_1)\dots(x - x_{n-1}) \quad \dots(5)$$

In equation (5), let us put $x - x_0 = uh$.

Then,
$$(x - x_1) = (x - x_0) - (x_1 - x_0) = uh - h = (u - 1)h$$

$$(x - x_2) = (x - x_1) - (x_2 - x_1)$$
$$= (u - 1)h - h = (u - 2)h$$

In general, $(x - x_n) = (u - n)h$.

Hence equation (5) becomes,

$$y(x) = P_n(x) = y_0 + \frac{\Delta y_0}{h} \cdot uh + \frac{\Delta^2 y_0}{2!\,h^2} \cdot uh.\,(u-1)h + \frac{\Delta^3 y_0}{3!\,h^3} \cdot uh \cdot (u-1)h \cdot (u-2)h + \ldots$$

i.e.,

$$y(x_0 + uh) = P_n(x_0 + uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots(6)$$

where $u = \dfrac{x - x_0}{h}$.

Equation (6) is known as Gregory-Newton forward difference formula.

Aliter. The above Newton's forward difference formula can be derives by symbolic operator methods.

Let $y(x) = P_n(x)$

Putting $x = x_0 + uh$.

Then
$$y(x) = P_n(x_0 + uh) = E^u P_n(x_0)$$
$$= E^u y_0 \qquad (\because P_n(x_0) = y(x_0) = y_0)$$
$$= (1 + \Delta)^u y_0 \qquad (\because E = 1 + \Delta)$$
$$= \left[1 + \binom{u}{1}\Delta + \binom{u}{2}\Delta^2 + \binom{u}{3}\Delta^3 + \ldots\right] y_0$$

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$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

where $u = \dfrac{x - x_0}{h}$.

Remark. The first two terms of the R.H.S. equation (6) give the result for linear interpolation polynomial. The first three terms give a parabolic interpolation and so on.

Note 1. If $y(x)$ is a polynomial of nth degree, then $\Delta^{n+1} y_0$ and other higher differences will be zero.

Note 2. Since Newton's forward interpolation formula utilises y_0 and the forward differences of y_0, it is used mainly for interpolating the values of y near the beginning of a set of tabular values.

Note 3. This formula is applicable only if the interval of differencing h is constant.

3. GREGORY-NEWTON BACKWARD INTERPOLATION FORMULA FOR EQUAL INTERVALS

Suppose we want to inter polate the values of y nearer to the end of a set of tabular values. In this case, Newton's forward interpolation formula can not be used. For this purpose, we get another backward interpolation formula.

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the values $x_0, x_1, x_2, \ldots x_n$ of the independent variable x. Let the values of x be at equidistant intervals.

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i.e., $x_i - x_{i-1} = h, i = 1, 2, \ldots n.$

Then $x_i = x_{i-1} + h, i = 1, 2, \ldots n.$

Also $x_i = x_0 + ih$ for $i = 1, 2, \ldots n$.

Let $y(x)$ be a polynomial of the nth degree in x taking the same values as y_x corresponding to the x values $x_0, x_1, x_2, \ldots x_n$.

i.e., $y(x_i) = y_i$ for $i = 1, 2, \ldots n$.

Now we want to find a collocation polynomial $P_n(x)$ of degree n in x such that

$$y(x_i) = P_n(x_i), i = 0, 1, 2, \ldots n.$$

Let
$$y(x) = P_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \ldots + a_n(x - x_n)(x - x_{n-1}) \ldots (x - x_1) \quad \ldots(1)$$

This polynomial contains the $(n + 1)$ constants $a_0, a_1, a_2, \ldots a_n$ which can be determined as follows:

Putting $x = x_n, x_{n-1}, \ldots x_0$ successively in (1), we get

$$y(x_n) = a_0 \Rightarrow y_n = a_0 \quad \ldots(2)$$

$$y(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

$$\Rightarrow y_{n-1} = y_n - a_1 h \quad (\because \text{ by (2) and } x_n - x_{n-1} = h)$$

$$\Rightarrow a_1 h = y_n - y_{n-1}$$

$$\Rightarrow a_1 = \frac{\nabla y_n}{1!\,h} \quad (\because y_n - y_{n-1} = \nabla y_n)$$

$$y(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\Rightarrow y_{n-2} = a_0 + a_1(\overline{x_{n-2} - x_{n-1}} + \overline{x_{n-1} - x_n}) + a_2(\overline{x_{n-2} - x_{n-1}} + \overline{x_{n-1} - x_n})(x_{n-2} - x_{n-1})$$

$$\Rightarrow y_{n-2} = y_n + \frac{\nabla y_n}{h}(-2h) + a_2(-2h)(-h)$$

$$\Rightarrow 2h^2 a_2 = y_{n-2} - y_n + 2\nabla y_n$$
$$= \overline{y_{n-2} - y_{n-1}} + \overline{y_{n-1} - y_n} + 2\nabla y_n$$
$$= -\nabla y_{n-1} - \nabla y_n + 2\nabla y_n$$
$$= \nabla y_n - \nabla y_{n-1} = \nabla^2 y_n$$

$$\therefore a_2 = \frac{\nabla^2 y_n}{2!\,h^2}.$$

Similarly $a_3 = \dfrac{\nabla^3 y_n}{3!\,h^3}$ and so on.

Substituting these values in (1), we get

$$y(x) = y_n + (x - x_n)\cdot\frac{\nabla y_n}{1!\,h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 y_n}{2!\,h^2} + (x - x_n)(x - x_{n-1})(x - x_{n-2})\frac{\nabla^3 y_n}{3!\,h^3} + \ldots \quad \ldots(3)$$

Put $x - x_n = vh \Rightarrow v = \dfrac{x - x_n}{h}$ in equation (3).

Then
$$x - x_{n-1} = x - x_n + x_n - x_{n-1}$$
$$= vh + h = (v+1)h$$
$$x - x_{n-2} = \overline{x - x_{n-1}} + \overline{x_{n-1} - x_{n-2}}$$
$$= (v+1)h + h = (v+2)h$$
$$\ldots\ldots\ldots\ldots$$
$$x - x_0 = x - x_{n-n} = (v+n)h$$

∴ Equation (3) becomes,

$$y(x_n + vh) = y_n + (vh) \cdot \frac{\nabla y_n}{1!\,h} + (vh) \cdot (v+1)h \cdot \frac{\nabla^2 y_n}{2!\,h^2} + (vh) \cdot (v+1)h \cdot (v+2)h \cdot \frac{\nabla^3 y_n}{3!\,h^3} + \ldots$$

i.e.,
$$y(x_n + vh) = y_n + \frac{v}{1!} \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n + \ldots \quad \ldots(4)$$

where $v = \dfrac{x - x_n}{h}$.

Equation (4) is known as Gregory-Newton backward difference formula.

Aliter. The above formula can be derived by symbolic operator methods.

Let $\quad y(x) = P_n(x)$

Putting $x = x_n + vh$

Then
$$y(x) = P_n(x_n + vh)$$
$$= E^v P_n(x_n)$$
$$= (E^{-1})^{-v} y_n \text{ where } P_n(x_n) = y_n$$
$$= (1 - \nabla)^{-v} y_n$$
$$= \left[1 + v\nabla + \frac{v(v+1)}{2!} \nabla^2 + \frac{v(v+1)(v+2)}{3!} \nabla^3 + \ldots\right] y_n$$

i.e.,
$$y(x) = y_n + \frac{v}{1!} \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n + \ldots$$

where $v = \dfrac{x - x_n}{h}$. **حفظ هذا قانون الطريقة الثالثة**

Note. This formula is used to interpolate the values of y nearer to the end of a set tabular values and also for extrapolating values of y a short distance ahead of y_n.

WORKED EXAMPLES

Example 1. *The following data give I, the indicated H.P. and v, the speed in knots developed by a ship.*

v:	*8*	*10*	*12*	*14*	*16*
I:	*1000*	*1900*	*3250*	*5400*	*8950*

Find I when v = 9, using Newton's interpolation formula.

Solution. We note that $v = 9$ is near the beginning of the table. Hence to get the corresponding I, we use Newton's forward interpolation formula. So we prepare the forward difference table.

طريقة الحل:

1.ناخذ قيمة X=V=9 من اي قيمة الى X تقترب في الجدول نحددها ثم نبدأ الحل.

2.حددنا X=8 في الجدول ناخذ قيم Y بحيث :

Y2 - Y1=1900 - 100=900

Y3 - Y2=3250 -1900=1350

وهكذا لبقية اعمدة الجدول نتوقف الى ان تبقى قيمة واحدة فقط.

3.نستخرج قيمة u بتطبيق القانون.

4. واخيرا نستخرج قيمة y بتطبيق القانون طريقة1 . اما الطريقة 2 تبدأ من اخر قيمة الى X

$x = v$	$y = I$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
8	1000				
		900			
10	1900		450		
		1350		350	
12	3250		800		250
		2150		600	
14	5400		1400		
		3550			
16	8950				

To find I $(v = 9)$

Here $h = 2$ and $u = \dfrac{9-8}{2} = \dfrac{1}{2}$

Newton's forward interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\therefore \quad I_{v=9} = 1000 + \frac{\left(\frac{1}{2}\right)}{1!}(900) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}(450) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!}(350) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{4!}(250)$$

$$= 1000 + 450 - 56.25 + 21.88 - 9.77 = 1405.86.$$

Example 2. *The following are data from the Steam Table:*

Temperature °C :	*140*	*150*	*160*	*170*	*180*
Pressure kgf/cm²:	*3.685*	*4.854*	*6.302*	*8.076*	*10.225*

Using Newton's formula, find the pressure of the steam for a temperature of 142°.

Solution. Here 142° is near the beginning of the table. Hence to get the corresponding pressure, we should use Newton's forward interpolation formula. So we prepare the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

Here $h = 10$ and $u = \dfrac{142-140}{10} = \dfrac{2}{10} = .2$

By using Newton's forward interpolation formula,

$$y(142) = 3.685 + (.2) \times 1.169 + \frac{(.2)(.2-1)}{2!} \times (0.279) + \frac{(.2)(.2-1)(.2-2)}{3!} \times (0.047) + \frac{(.2)(.2-1)(.2-2)(.2-3)}{4!} \times (0.002)$$

$$= 3.685 + 0.2338 - 0.0223 + 0.0022 - 0.0001$$

$$= 3.8986.$$

Example 3. *From the following table, find tan 45°15′.*

$x°$:	45	46	47	48	49	50
$\tan x°$:	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

Solution. Here $x = 45°\ 15'$ is near the beginning of the table. We prepare the forward difference table.

x	$y = \tan x°$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
45	1.00000					
		0.03553				
46	1.03553		0.00131			
		0.03684		0.00009		
47	1.07237		0.00140		0.00003	
		0.03824		0.00012		−0.00005
48	1.11061		0.00152		−0.00002	
		0.03976		0.00010		
49	1.15037		0.00162			
		0.04138				
50	1.19175					

Here $h = 1°$ and $u = \dfrac{45°15' - 45°}{1°} = 0.25$

By Newton's forward interpolation formula

$$y(45°15') = 1.00000 + \frac{(0.25)}{1!}(0.03553) + \frac{(0.25)(0.25-1)}{2!}(0.00131) + \frac{(0.25)(0.25-1)(0.25-2)}{3!}(0.00009)$$

$$+ \frac{(0.25)(0.25-1)(0.25-2)(0.25-3)}{4!}(0.00003) + \frac{(0.25)(0.25-1)(0.25-2)(0.25-3)(0.25-4)}{5!}(-0.00005)$$

$$= 1.00000 + 0.00888 - 0.00012 + 0.00001 + 0.00000 \ldots = 1.00875.$$

Example 4. *Using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data:*

Hence find $f(2)$.

x:	0	5	10	15
$f(x)$:	14	379	1444	3584

Solution. Here $x_0 = 0$; $h = 5$ and $u = \frac{x-0}{5} = \frac{x}{5}$

We prepare the forward difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	14			
		365		
5	379		700	
		1065		375
10	1444		1075	
		2140		
15	3584			

Using Newton's forward interpolation formula,

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \ldots$$

i.e.,
$$f(x) = 14 + \frac{x}{5}(365) + \frac{1}{2}\left(\frac{x}{5}\right)\left(\frac{x}{5}-1\right)700 + \frac{1}{6}\left(\frac{x}{5}\right)\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)(375)$$

i.e.,
$$f(x) = 14 + 73x + \frac{x^2-5x}{14} + \frac{1}{2}(x)(x-5)(x-10)$$

i.e.,
$$f(x) = 14 + 73x + \frac{x^2-5x}{14} + \frac{1}{2}(x^3 - 15x^2 + 50x)$$

i.e.,
$$f(x) = \frac{x^3}{2} - \frac{104}{14}x^2 + \frac{1367}{14}x + 14$$

$\therefore$
$$f(2) = \frac{8}{2} - \frac{104}{14}(4) + \frac{1367}{14}(2) + 14 = 183.6.$$

Here $h = 10$ and $v = \dfrac{84-90}{10} = -0.6$

$$\therefore \qquad t_{p=84} = 304 + \frac{(-0.6)}{1!}(28) + \frac{(-0.6)(-0.6+1)}{2!}(2)$$

$$= 304 - 16.8 - 0.24 = 286.96.$$

Example 6. *The hourly declination of the moon on a day is given below. Find the declination at $3^h\,35^m\,15^s$ and 5^h.*

Hour:	*0*	*1*	*2*	*3*	*4*
Decli:	*8°29′53.7″*	*8°18′19.4″*	*8°6′43.5″*	*7°55′6.1″*	*7°43′27.2″*

Solution. We prepare the difference table.

في هذا السؤال نستخدم الطريقة 2 وعرفناها من خلال قيمة X=3 في السؤال مقارنتها مع قيم x في الجدول تكون اوخر القيم X.
1.نقلب دالتا Y في الجدول.
2.نجد قيم اعمدة الجدول بنفس الطريقة السابقة.
3. نجدh= x2-x1 ونعوضها بالقانون لايجاد قيمة u.
4.نجد قيمة y بقانون طريقة2.

Hour (x)	Decli (y)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	8°29′53.7″				
		−11′34.3″			
1	8°18′19.4″		−1.6″		
		−11′35.9″		0.1″	
2	8°6′19.4″		−1.5″		−0.1″
		−11′37.4″		0.0″	
3	7°55′6.1″		−1.5″		
		−11′38.9″			
4	7°43′27.2″				

Here $3^h\,35^m\,15^s$ and 5^h are near the end of the table value. So we use Newton's Backward interpolation formula.

Let $x = 3^h\,35^m\,15^s$. Then $v = \dfrac{3h\,35m\,15s - 4h}{1h} = \dfrac{-0h\,24m\,45s}{1h}$

$$= -\frac{1485s}{3600s} = -0.4125.$$

By Newton's backward interpolation formula,

$$y(3h\,35m\,15s) = 7°43'27.2'' + (-0.4125)(-11'38.9'') + \frac{(-0.4125)(0.5875)}{2}(-1.5'')$$

$$= 7°43'27.2'' + 4'48.29'' + 0.18''$$

$$= 7°48'16''.$$

Let $x = 5h$. Then $v = \dfrac{5-4}{1} = 1$

Hence $$y(x=5) = 7°43'27.2'' + (1)(-11'38.9'') + \frac{(1)(2)}{2}(-1.5'') + \ldots$$

$$= 7°31'46.8''.$$

EXERCISES

1. Estimate y when $x = 47$ and $x = 63$ from the following data:

x:	45	50	55	60	65
y:	114.84	96.16	83.32	74.48	68.48

[Ans. 106.5, 68.4]

2. Estimate the values of y at $x = 21$ and $x = 28$ from the following data:

x:	20	23	26	29
y:	0.3420	0.3907	0.4384	0.4848

[Ans. 0.3583, 0.4695]

3. Estimate y when $x = 42$ from the following data:

x:	20	25	30	35	40	45
y:	354	332	291	260	231	204

[Ans. 218.7]

4. Find the value of $f(1.02)$ from the following data:

x :	1.0	1.1	1.2	1.3	1.4
$f(x)$:	1.841	1.891	0.932	0.964	0.985

[Ans. 1.851]

5. Find $y(1.02)$ from the following data:

x:	1.00	1.05	1.10	1.15	1.20
y:	0.3413	0.3531	0.3643	0.3749	0.3849

[Ans. 0.3461]

6. From the following table find tan 0.12 and tan 0.26.

x :	0.10	0.15	0.20	0.25	0.30
$\tan x$:	0.1003	0.1511	0.2027	0.2553	0.3093

[Ans. 0.1205, 0.2662]

7. Find y at $x = 3$ and $x = 4$

x:	2	6	10	14
y:	40.2	42.4	51.0	72.4

[Ans. 40.5, 40.9]

8. Given the following data, express θ as function of t

t:	0	1	2	3	4
θ:	3	6	11	18	27

[Ans. $\theta = t^2 + 2t + 3$]

9. Find $y(12)$

x:	10	15	20	25	30	35
y:	35.3	32.4	29.2	26.1	23.2	20.5

[Ans. 34.22]